Quantum Physics 1 - Test 1 with solutions

A particle of mass m is in the state

$$\Psi = Axe^{-a(mx^2/\hbar+3it)}, \tag{1}$$

where a is some positive real constant and $A = 2\left(\frac{2ma}{\pi\hbar}\right)^{1/4}\sqrt{\frac{ma}{\hbar}}$.

a. For what potential energy $V(x)$ does Ψ satisfy the Schrödinger equation? (*5 points*)
Hint: the Schrödinger equation is given by $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$

Solution:
Filling in the Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi \tag{2}$$

$$i\hbar(-3ia)Axe^{-a(mx^2/\hbar+3it)} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(A\frac{-2am}{\hbar}x^2e^{-a(mx^2/\hbar+3it)} + Ae^{-a(mx^2/\hbar+3it)}\right) + V(x)\Psi \tag{3}$$

$$3a\hbar Axe^{-a(mx^2/\hbar+it)} = -\frac{\hbar^2}{2m}\left(A\left(\frac{-2am}{\hbar}\right)^2 x^3e^{-a(mx^2/\hbar+3it)} + \frac{-6am}{\hbar}Axe^{-a(mx^2/\hbar+3it)}\right) + V(x)\Psi \tag{4}$$

$$3a\hbar\Psi = -\frac{\hbar^2}{2m}\left(\left(\frac{-2am}{\hbar}\right)^2 x^2\Psi + \frac{-6am}{\hbar}\Psi\right) + V(x)\Psi \tag{5}$$

$$3a\hbar\Psi = -2ma^2x^2\Psi + 3a\hbar\Psi + V(x)\Psi \tag{6}$$

So $V(x) = 2ma^2x^2$.

b. What are the expectation values $\langle x\rangle$ and $\langle p\rangle$? Explain your answer. (*4 points*)
Hint: you do not have to work out any integrals.

Solution:
$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2 = 0$, because $x|\Psi|^2$ is an odd function that is integrated along an even interval.
Because of Ehrenfest's theorem, $\langle p\rangle = m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = 0$.

Quantum Physics 1 - Test 2 with solutions

Consider a particle of mass m subject to the harmonic oscillator potential ($V(x) = \frac{1}{2}m\omega^2 x^2$), and assume that, at $t = 0$, the particle is in the state

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\left(\psi_0(x) + \psi_1(x)\right).$$

a. (*1 pts*) Add the time dependence to Ψ (i.e., find an expression for $\Psi(x,t)$).
Solution. We have

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(e^{-iE_0 t/\hbar}\psi_0(x) + e^{-iE_1 t/\hbar}\psi_1(x)\right).$$

b. (*3 pts*) Calculate $\langle x \rangle$. (*Hint*: use orthonormality to evaluate integrals, and consult the hints at the bottom.)
Solution. Define $f(t) = \exp(-iE_0 t/\hbar)$ and $g(t) = \exp(-iE_1 t/\hbar)$. Then, we have (exploiting orthonormality)

$$\begin{aligned}
\langle x \rangle &= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\int (f\psi_0 + g\psi_1)^*(a_+ + a_-)(f\psi_0 + g\psi_1)\mathrm{d}x \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\int (f\psi_0 + g\psi_1)^*(fa_+\psi_0 + fa_-\psi_0 + ga_+\psi_1 + ga_-\psi_1)\mathrm{d}x \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\int (f\psi_0 + g\psi_1)^*(f\psi_1 + g\psi_0)\mathrm{d}x \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\int (f^*g\psi_0^*\psi_0 + fg^*\psi_1^*\psi_1)\mathrm{d}x \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(f^*g + fg^*) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,\mathrm{Re}\{f^*g\} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos\left((E_1 - E_0)t/\hbar\right) = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t).
\end{aligned}$$

c. (*3 pts*) Calculate $\langle x^2 \rangle$ and σ_x.

Solution. First, we note that $a_+a_-\psi_n = n\psi_n$ and $a_-a_+\psi_n = (n+1)\psi_n$. We have (exploiting orthonormality and the fact that squares of ladder operators in this case yield cancelling terms)

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \frac{1}{2} \cdot \frac{\hbar}{2m\omega} \int (f\psi_0 + g\psi_1)^* (a_+ + a_-)^2 (f\psi_0 + g\psi_1) \mathrm{d}x \\
&= \frac{\hbar}{4m\omega} \int (f\psi_0 + g\psi_1)^* (a_+^2 + a_+a_- + a_-a_+ + a_-^2)(f\psi_0 + g\psi_1) \mathrm{d}x \\
&= \frac{\hbar}{4m\omega} \int (f\psi_0 + g\psi_1)^* (a_+a_- + a_-a_+)(f\psi_0 + g\psi_1) \mathrm{d}x \\
&= \frac{\hbar}{4m\omega} \int (f\psi_0 + g\psi_1)^* (f \cdot (a_+a_- + a_-a_+)\psi_0 + g \cdot (a_+a_- + a_-a_+)\psi_1) \mathrm{d}x \\
&= \frac{\hbar}{4m\omega} \int (f\psi_0 + g\psi_1)^* (f\psi_0 + 3g\psi_1) \mathrm{d}x \\
&= \frac{\hbar}{4m\omega} \int (f^* f \psi_0^* \psi_0 + 3g^* g \psi_1^* \psi_1) \mathrm{d}x \\
&= \frac{\hbar}{4m\omega} \left(|f|^2 + 3|g|^2 \right) \\
&= \frac{\hbar}{4m\omega} (1+3) = \frac{\hbar}{m\omega}.
\end{aligned}$$

By $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$, we have

$$\sigma_x = \sqrt{\frac{\hbar(2 - \cos^2(\omega t))}{2m\omega}}.$$

d. (*2 pts*) Is there a moment in time when the momentum standard deviation σ_p can be zero? Explain your answer.

Heisenberg's uncertainty principle states that $\sigma_x \sigma_p \geq \hbar/2$. It follows that σ_p cannot be zero.

Hint: x and p can be written in terms of ladder operators as follows:

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-), \, p = i\sqrt{\frac{\hbar m\omega}{2}} (a_+ - a_-).$$

We also have the following ladder operator relations: $a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$, $a_-\psi_n = \sqrt{n}\,\psi_{n-1}$.

Quantum Physics 1 - Test 3 Solutions

Consider the *moving* delta-function well, whose potential and normalized solution to the time-dependent Schrödinger equation are given by

$$V(x,t) = -\alpha\delta(x - vt)$$
$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar}\exp\bigl(-m\alpha|x-vt|/\hbar^2\bigr)\exp\bigl(-i\bigl[\bigl(E+\tfrac{1}{2}mv^2\bigr)t - mvx\bigr]/\hbar\bigr)$$

where v is the (constant) velocity of the well, $\alpha > 0$, $\exp(a) = e^a$ and $E = -m\alpha^2/2\hbar^2$.

a. *(3p)* Calculate $\langle x\rangle$. *Hint*: You can use the integral $\int_0^\infty x^n e^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}}$.
Solution:

$$\begin{aligned}
\langle x\rangle &= \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty} x\exp\bigl(-2m\alpha|x-vt|/\hbar^2\bigr)\,\mathrm{d}x \qquad \text{substitute } y = x - vt\\
&= \frac{m\alpha}{\hbar^2}\left(\int_{-\infty}^{\infty} y\exp\bigl(-2m\alpha|y|/\hbar^2\bigr)\,\mathrm{d}y + vt\int_{-\infty}^{\infty}\exp\bigl(-2m\alpha|y|/\hbar^2\bigr)\,\mathrm{d}y\right)\\
&= \frac{m\alpha}{\hbar^2}\left(2vt\int_0^\infty \exp\bigl(-2m\alpha y/\hbar^2\bigr)\,\mathrm{d}y + \int_0^\infty y\exp\bigl(-2m\alpha y/\hbar^2\bigr)\,\mathrm{d}y - \int_0^\infty y\exp(2m\alpha y/\hbar)\,\mathrm{d}y\right)\\
&= \frac{m\alpha}{\hbar^2}\left(2vt\frac{\hbar^2}{2m\alpha} + \frac{\hbar^4}{4m^2\alpha^2} - \frac{\hbar^4}{4m^2\alpha^2}\right) = vt
\end{aligned}$$

b. *(2p)* What is the probability to find the particle to either side of the well?
Solution: Because Ψ is centered at the well ($\langle x\rangle = vt$), you have a 50% chance to find the particle at either side of the well.

c. *(1p)* Calculate $\langle p\rangle$. **Solution:** $\langle p\rangle = m\,\mathrm{d}\langle x\rangle/\mathrm{d}t = mv$.

d. *(3p)* Recall the equation for the probability current

$$J \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right).$$

Calculate the probability current for this wave function. Which direction does the probability current flow? *(1p) Bonus:* Express J in terms of Ψ
Hint: To avoid having to differentiate the absolute value, write $\Psi = cf(x,t)g(x,t)$ with each f, g containing one exponent, such that $f = f^*$ and write out J before doing any differentiation.

Solution: We define $\Psi(x,t) = cf(x,t)g(x,t)$ where f is the first exponent and g is the second. This leads to $f^* = f$, $gg^* = 1$, $g' = (i/\hbar)mv\cdot g$ and $(g^*)' = -(i/\hbar)mv\cdot g^*$.

$$\begin{aligned}
J &\equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)\\
&= \frac{i\hbar}{2m}\left(cfg\cdot c(f'g^* + f(g^*)') - cfg^*\cdot c(f'g + fg')\right)\\
&= \frac{i\hbar}{2m}\left(c^2ff'gg^* + c^2f^2g(g^*)' - c^2ff'gg^* - c^2f^2g'g^*\right)\\
&= \frac{i\hbar}{2m}c^2f^2\bigl(g(g^*)' - g'g^*\bigr) = \frac{i\hbar}{2m}c^2f^2\left(-\frac{i}{\hbar}mvgg^* - \frac{i}{\hbar}mvgg^*\right)\\
&= \frac{mv\alpha}{\hbar^2}\exp\bigl(-2m\alpha|x-vt|/\hbar^2\bigr) = v|\Psi|^2
\end{aligned}$$

The current flows to the right (positive x).

Quantum Physics test 4

Consider the following potential: $V = 0$ for $x < 0$ and $V = -V_c$ for $x > 0$, where V_c is a positive constant. For a particle moving to the right with energy $E_0 > 0$:

1. Sketch the potential and write the solution of the eigenvalue equation $\hat{H}\psi = E\psi$ for $x < 0$ and $x > 0$, considering no incoming wave from the right (3 points).

   The solution is a superposition of plane waves

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & , x < 0 \\ Fe^{ik_2x} & , x > 0 \end{cases}$$

   where $k_1 = \sqrt{2mE_0}/\hbar$ and $k_2 = \sqrt{2m(E_0 + V_c)}/\hbar$.

2. Find the reflection coefficient R in terms of E_0 and V_c (3 points).

   R is defined as $R = |B|^2/|A|^2$. Continuity of the wavefunction and its derivative at $x = 0$ gives

$$\begin{cases} A + B = F \\ k_1(A - B) = k_2F \end{cases} \quad \Rightarrow (k_2 - k_1)A = (k_2 + k_1)B\,,$$

   and therefore R is equal to

$$R = \left(\frac{k_2 - k_1}{k_2 + k_1}\right)^2 = \frac{(\sqrt{V_c + E_0} - \sqrt{E_0})^2}{(\sqrt{V_c + E_0} + \sqrt{E_0})^2}\,.$$

3. Verify that R and T (transmission coefficient) sum up to 1. T is given by the formula

$$T = \sqrt{\frac{E_0 + V_c}{E_0}}\frac{|F|^2}{|A|^2}\,,$$

   with A and F the coefficients of the plane wave travelling to the right for $x < 0$ and $x > 0$ accordingly (4 points).

$$2A = \frac{k_1 + k_2}{k_1}F \Rightarrow \frac{|F|^2}{|A|^2} = \frac{4k_1^2}{(k_1 + k_2)^2} = \frac{4E_0}{(\sqrt{V_c + E_0} + \sqrt{E_0})^2}\,,$$

   and so

$$T = \sqrt{\frac{E_0 + V_c}{E_0}}\frac{4E_0}{(\sqrt{V_c + E_0} + \sqrt{E_0})^2}\,,$$

   $T + R = 1$ is satisfied.

Test 5 Quantum Physics 1

a) Using the formula:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2, \tag{1}$$

show that $\sigma_H \sigma_p \geq \frac{\hbar}{2} \langle \frac{dV}{dx} \rangle$. (3 points)

Solution: Employ a test function $g(x)$

$$\begin{aligned}
[H, p]g(x) &= [V(x), -i\hbar \frac{\partial}{\partial x}]g(x) = -i\hbar V(x) \frac{\partial}{\partial x} g(x) + i\hbar \frac{\partial}{\partial x}(V(x)g(x)) \\
&= -i\hbar V(x) \frac{\partial}{\partial x} g(x) + i\hbar V(x) \frac{\partial}{\partial x} g(x) + i\hbar \frac{\partial}{\partial x}(V(x))g(x) \\
&= i\hbar \frac{\partial}{\partial x}(V(x))g(x)
\end{aligned}$$

Then plug in the result.

b) Consider the harmonic oscillator ($V(x) = \frac{1}{2} m\omega^2 x^2$). Does the uncertainty relation derived above give you information about the ground state? What does it say about the excited states?

Solution: The ground state and excited states are stationary states, so the uncertainty relation tells you $< V'(x) > \propto < x >= 0$. This is because all stationary states are either odd or even.

c) A generic state of the harmonic oscillator is a superposition of the ground state and all the excited states. What does the uncertainty relation tell you about generic states? (3 points)

Solution: A generic state is not a stationary state, so the left-hand side of the uncertainty relation becomes non-zero. This means $< x >$ can obtain a non-zero expectation value. Even though all eigenstates are even or odd, the superposition of them does not need to be. The uncertainty principle now relates the three quantities σ_H, σ_p and $< x >$.

Quantum Physics 1 - Test 6

Consider the ground state of hydrogen, of which the wave function is given by:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}.$$

a) *(4p)* Calculate $\langle V \rangle$. *Hint:* Use the fact that $\langle V \rangle = -\frac{\hbar^2}{ma}\left\langle \frac{1}{r} \right\rangle$, and use integration by parts.

Solution:

$$\begin{aligned}
\left\langle \frac{1}{r} \right\rangle &= \frac{1}{\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{1}{r} e^{-2r/a} r^2 \sin\phi \,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{4}{a^3} \int_0^{\infty} r e^{-2r/a}\,\mathrm{d}r = \frac{4}{a^3}\left(\frac{-a}{2} r e^{-2r/a} \Big|_0^{\infty} - \frac{-a}{2} \int_0^{\infty} e^{-2r/a}\,\mathrm{d}r \right) \\
&= \frac{2}{a^2} \int_0^{\infty} e^{-2r/a}\,\mathrm{d}r = \frac{2}{a^2} \frac{-a}{2}\, e^{-2r/a} \Big|_0^{\infty} = \frac{1}{a} \\
\langle V \rangle &= -\frac{\hbar^2}{ma^2}
\end{aligned}$$

b) *(3p)* Using your result of (a), calculate $\langle \mathbf{p}^2 \rangle$. If you did not manage to complete (a), find $\langle \mathbf{p}^2 \rangle$ in terms of $\langle V \rangle$. The energy of the ground state ψ_{100} is given by

$$E_1 = -\frac{\hbar^2}{2ma^2}.$$

Solution:
We know that $H\psi_{100} = E_1\psi_{100}$ and $H = \frac{p^2}{2m} + V$, so we can combine the two into $\langle \mathbf{p}^2 \rangle = 2m(E_1 - \langle V \rangle)$.

$$\left\langle \mathbf{p}^2 \right\rangle = -2m\left(\frac{\hbar^2}{2ma^2} - \frac{\hbar^2}{ma^2} \right) = -\frac{2\hbar^2}{a^2}\left(\frac{1}{2} - 1 \right) = \frac{\hbar^2}{a^2}$$

c) *(2p)* What is $\langle {p_x}^2 \rangle$? *Hint:* You do not have to do any long calculations.
Solution: Since $\langle \mathbf{p}^2 \rangle = \langle {p_x}^2 \rangle + \langle {p_y}^2 \rangle + \langle {p_z}^2 \rangle$ and since the wave function is only dependent on r (not ϕ and θ), we have that $\langle {p_x}^2 \rangle = \frac{1}{3} \langle \mathbf{p}^2 \rangle = \frac{\hbar^2}{3a^2}$.

Quantum Physics 1 - Test 7

Two particles, each of mass m, are attached to the ends of a massless rigid rod of length a. This system (called a *rigid rotor*) is free to rotate in all three dimensions about the fixed center point. The classical energy of this system is given by $E = L^2/(2I)$, where $I = ma^2/2$ is the moment of inertia of the system.

Now, we consider the case that a is very small and hence we will describe this system quantum mechanically. Thus, we will use operators, resulting in the Hamiltonian below:

$$\hat{H} = \frac{\hat{L}^2}{2I}.$$

(a) (*2 pts*) Find the allowed energies of the rigid rotor (i.e., find the eigenvalues of $\hat{H}$).
Solution. The eigenvalues of $\hat{L}^2$ are known: $\hbar^2 l(l+1)$, so we have

$$E_n = \frac{\hbar^2 n(n+1)}{2I} = \frac{\hbar^2 n(n+1)}{ma^2}$$

where we label the energies with n ($n = 0,\ 1,\ 2,\ \ldots$) rather than with l.

(b) (*2 pts*) Find the corresponding degeneracies.
Solution. Given l corresponding to the eigenfunctions f_l^m, the degeneracy is equal to the number of different eigenfunctions f_l^m which is equal to the number of different values m can attain. This number is, as always, equal to $2l+1$. This means that the degeneracy corresponding to E_n is equal to $2n+1$, for all n.

(c) (*3 pts*) Construct the ground state of the rigid rotor.
Solution. The normalized eigenfunctions corresponding to energy E_n are given by the spherical harmonics $Y_n^m(\theta, \phi)$ ($m = -n, -n+1, \ldots, n-1, n$). In this case, $n = 0$ so $m = 0$ and hence the ground state is $Y_0^0(\theta, \phi)$, which is given to be

$$Y_0^0(\theta, \phi) = \frac{1}{2\sqrt{\pi}}.$$

(d) (*2 pts*) Is there a difference between the classical ground state (i.e., the lowest possible value for the classical energy) and the quantum mechanical ground state energy? Explain how this is compatible with Heisenberg's uncertainty principle.
Solution. Classically, the lowest possible value for the energy is simply zero. The energy corresponding to the quantum mechanical ground state is $E_0 = 0$. This means that there is no difference. Furthermore, Heisenberg always places a constraint on two quantum numbers. In the harmonic oscillator, for example, to obtain an energy equal to zero, one needs to set both x and p equal to zero which is impossible due to Heisenberg's uncertainty principle. In the case of the rigid rotor, we can simply set the energy equal to zero by just taking our quantum number n to be zero.

Test 8 Quantum Physics 1

a) Consider a system of two non-interacting identical particles, one in state ψ_a and one in state ψ_b. These states are orthogonal and normalized. Show that:

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2$$

where $\langle x \rangle_{ab} = \int x\psi_a^\star(x)\psi_b(x)dx$. This term indicates the *exchange force*. It takes the upper sign in $\mp$ for bosons and the lower for fermions.
Solution: See section 5.1.2 of the Griffiths book for the full solution.

b) Consider two non-interacting identical *bosons* in the one-dimensional harmonic oscillator potential, $V = \frac{1}{2}m\omega^2 x^2$. The ground state for a single particle in this potential is:

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Is there an exchange force in the ground state of the two-particle system? If so, is it attractive or repulsive? Assume the particles are in the same spin state.

Solution: When the particles are in the same spin state, the spin part of the wavefunction is symmetric under exchange. Since the overall wavefunction is symmetric, the spatial wavefunction is symmetric as well. The ground state is therefore simply $\Psi_0(x_1, x_2) = \psi_0(x_1)\psi_0(x_2)$. Then $\langle x \rangle_{ab} = \int x\psi_0^\star(x)\psi_0(x)dx = 0$, since the integrand is odd. There is no exchange force.

c) Consider two non-interacting identical *fermions* in the same potential. However, this time the particles are in the singlet state where the total spin is equal to zero. Is there an exchange force in the ground state of this system? If so, is it attractive or repulsive?

Solution: This time the spin part is *anti-symmetric* under exchange. This means that the spatial wavefunction is *symmetric* under exchange, even though the particles are fermions. Again, the spatial part of the ground state wavefunction is $\Psi_0(x_1, x_2) = \psi_0(x_1)\psi_0(x_2)$, so there is no exchange force.